Chapter 13

Extensions

EXTENSION 13E1: QUANTITATIVE REPEATED FACTORS

In this extension, we make a point about contrasts in repeated factors that is pertinent when the repeated factor is quantitative, such as time or dosage level. There is a special meaning that can be attached to a subject's linear D score because each subject's linear D score is proportional to the slope of that subject's least-squares regression line when the score is regressed on the quantitative predictor variable. To illustrate this point, we again consider the data of Table 13.2 in the book. Let's suppose that the three time periods represented here are 12 months, 24 months, and 36 months. What would happen if we performed a regression analysis of the three scores for subject 1? There are 3 X values: 12, 24, and 36. There are three corresponding Y values: 2, 3, and 5. The slope of a least-squares regression line is given by

$$b = \frac{\Sigma(X - \bar{X})(Y - \bar{Y})}{\Sigma(X - \bar{X})^2}.$$

Substituting the three X and Y pairs into this formula results in a value of $b = .125$. Thus the best-fitting regression line (in the sense of least squares) suggests that this subject's score increases by one-eighth of a point every month. Table 13E1.1 shows the slope of the least-squares regression

Table 13E1.1
Values of the slope of the least-squares regression line and of the linear D variable for data of Table 13.2

Subject	***Slope***	***Linear D***
1	.125	3
2	.208	5
3	.083	2
4	.000	0
5	.208	5
6	.250	6
7	.083	2
8	.042	1

Material to accompany *Designing Experiments and Analyzing Data: A Model Comparison Perspective*, Third Edition by Scott E. Maxwell, Harold D. Delaney, and Ken Kelley (2018)

line for each of the eight subjects, as well as the score on the linear D variable, reproduced from Table 13.13 in the book. There is a striking relationship between the numbers in the two columns of Table 13E1.1. Every subject's score on D is 24 times his or her slope.[1] In general, it can be shown that the slope for the ith subject, b_i, is related to the subject's D score, D_i, by the following formula:

$$b_i = D_i \,/\, h\sum_{j=1}^{a} c_j^2.$$

The value of h expresses the relationship between the units of the time factor (expressed as deviations from the mean) and the values of the coefficients. Specifically, the two sets of numbers are necessarily proportional to one another, and h is the constant of proportionality. In our example, the units of the factor are 12, 24, and 36. Expressed as deviations from the mean, they become −12, 0, and 12 (because 24 is the mean of 12, 24, and 36). The coefficients for D are −1, 0, and 1. Thus $h = 12$ because the units of the factor are uniformly 12 times larger than the coefficients. Notice also that the sum of squared coefficients $\sum_{j=1}^{a} c_j^2$ equals 2, so we have

$$b_i = D_i \,/\, 24$$

in agreement with Table 13E1.1. We elaborate on this point because within-subjects designs often involve time as a within-subjects factor. Our illustration, which holds for any value of a, shows an interesting way in which the test of a linear trend across time can be conceptualized. In effect, such a test asks whether the average (i.e., mean) subject has scores whose straight-line slope across time equals zero. There is sometimes also a practical advantage to viewing the test of the linear trend from this perspective. If data are missing (presumably at random) for some mixture of subjects at varying time points, it is still possible to calculate a regression slope for each subject (as long as at least two observations are available for the subject). These slopes can then be used as D variables in a repeated-measures analysis.[2]

EXTENSION 13E2: FINDING D_{MAX}

Finding the coefficients of the D_{max} variable is more difficult than in the between-subjects design. In the between-subjects design, the optimal coefficients depend only on the sample means; however, in the within-subjects design, the optimal coefficients depend not only on the sample means but also on the interrelationships among the variables. It is necessary to use matrix algebra to incorporate these interrelationships into the calculation of the optimal coefficients. For this reason, we provide only a brief overview.

Let's return to the D variables of Table 13.3, which were obtained from the data in Table 13.2. When a multivariate analysis is performed, it is possible to obtain raw discriminant weights, which convey information about the relative weights to be assigned to variables so as to maximize an effect. For our data, the weight for D_1 is 1.08448, and the weight for D_2 is .19424. Thus D_1 is more influential than D_2 in rejecting the null hypothesis. However, our real interest is to find weights for the original Y variables. We can accomplish this through matrix multiplication. The discriminant weights must be written as a column vector, which we label **w**. For our data, we have

$$\mathbf{w} = \begin{bmatrix} 1.08448 \\ 0.19424 \end{bmatrix}.$$

Next, the coefficients used to derive the D variables from the original Y variables must be written in matrix form; we denote the matrix **T**. Each column corresponds to a D variable and each row to a Y variable. Recall that D_1 in Table 13.3 was defined as $D_1 = Y_2 - Y_1$. This implies that the first column of **T** has elements −1, 1, and 0. Similarly, D_2 was defined as $D_2 = Y_3 - Y_2$, so the second column of **T** has elements 0, −1, and 1. Combining the two columns yields

$$\mathbf{T} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

The coefficients for the maximum contrast (or optimal subeffect, as it is sometimes called) are obtained by multiplying **T** by **w**. Specifically, if we let **v** be the vector of optimal weights, then **v** = **Tw**. For our data,

$$\mathbf{v} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1.08448 \\ 0.19424 \end{bmatrix},$$

which results in

$$\mathbf{v} = \begin{bmatrix} -1.08448 \\ 0.89024 \\ 0.19424 \end{bmatrix}.$$

Thus the optimal D variable is

$$D_{\text{max}} = -1.08448 Y_1 + 0.89024 Y_2 + 0.19424 Y_3.$$

Several points must be made here. First, notice that the sum of the coefficients for D_{max} equals zero, as it must if the contrast is to be meaningful. Second, it can be shown that the observed F value for testing a null hypothesis that the population mean of D_{max} is zero equals 44.678. Recall that the omnibus observed F value for these data was 19.148. Thus, within rounding error, it is the case that

$$F_{\text{max}} = \frac{(n-1)(a-1)F_{\text{omnibus}}}{n-a+1}$$

because

$$44.678 = \frac{(8-1)(3-1)(19.148)}{8-3+1}.$$

Third, notice that the optimal coefficients do not closely match the pattern of mean differences for Y_1, Y_2, and Y_3. As we saw in Table 13.2, the means of Y_1, Y_2, and Y_3 are 6, 8, and 9, respectively.

Such a pattern would seem to suggest that the optimal contrast might weight the first and third levels most heavily. In fact, however, the optimal contrast is essentially a comparison of levels 1 and 2, with relatively little weight placed on the third level. The coefficients do not closely match the mean differences here because the optimal coefficients depend on the relationships among the variables and the standard deviations of the variables, as well as on the means. For our data, the correlation between Y_1 and Y_2 is much higher (r = .966) than either the correlation of Y_1 and Y_3 (r = .719) or Y_2 and Y_3 (r = .772). The standard deviations of the three variables are roughly the same. Because Y_1 and Y_2 are so highly correlated, the $Y_2 - Y_1$ difference score has a small variance. Another look at Equation 13.6 for testing a contrast shows that, all other things being equal, a small variance implies a large F value:

$$F = n\bar{D}^2 / s_D^2. \qquad \text{(13.6, repeated)}$$

Thus even though $Y_3 - Y_1$ has a larger mean than does $Y_2 - Y_1$, the F value for $Y_2 - Y_1$ is larger than the F value for $Y_3 - Y_1$ because the variance of $Y_2 - Y_1$ is so much smaller than the variance of $Y_3 - Y_1$.

We emphasize that rarely if ever would a researcher interpret the $D_{\max}$ variable itself. In our example, it is difficult to describe the psychological importance of $-1.08448Y_1 + .89024Y_2 + .19424Y_3$. However, the coefficients of $D_{\max}$ serve as a suggestion for more easily interpreted and hence more meaningful coefficients that might also prove to be significant. In our example, a natural choice would seem to be $D = Y_2 - Y_1$, which yields an F value of 37.3333 and hence easily exceeds the Roy–Bose critical value of 11.99.

EXTENSION 13E3: RECONCEPTUALIZATION OF ε IN TERMS OF E*(F)

We saw in the Chapter 13 discussion of "The Relationship Between the Multivariate Approach and the Mixed-Model Approach," in particular in the section on "Comparison of the Two Approaches," one difference between the multivariate approach and the mixed-model approach is that the multivariate approach is based on all elements of the **E*(F)** matrix, whereas the mixed-model approach uses only the diagonal elements of **E*(F)**. When homogeneity (i.e., sphericity) holds, only the diagonal elements are relevant, but when homogeneity does not hold, it is a mistake to ignore the off-diagonal elements of the **E*(F)** matrix.

One perspective on understanding how the ε-adjusted tests improve on the unadjusted mixed-model test is to realize that the ε-adjusted tests incorporate information about the off-diagonal elements of the **E*(F)** matrix into their calculation. Huynh (1978) showed that $\hat{\varepsilon}$ can be calculated from the elements of the **E*(F)** matrix in the following manner:

$$\hat{\varepsilon} = \frac{\left(\sum_{i=1}^{a-1} E_{ii}^{*}(\mathrm{F})\right)^2}{(a-1)\sum_{i=1}^{a-1}\sum_{j=1}^{a-1}(E_{ij}^{*}(\mathrm{F}))^2}, \qquad \text{(13E3.1)}$$

where $E_{ij}^{*}(\mathrm{F})$ refers to the element in row i and column j of the **E*(F)** matrix. Before proceeding to discuss the theoretical implications of this formula, it may be helpful to demonstrate its use on the data from Table 13.2. We saw in the book that **E*(F)** for these data is given by

$$\mathbf{E}^*(\mathbf{F}) = \begin{bmatrix} 16.0000 & -7.5055 \\ -7.5055 & 7.3333 \end{bmatrix}.$$

It then follows that the numerator of $\hat{\varepsilon}$ equals

$$\left(\sum_{i=1}^{a-1} E_{ii}^*(\mathrm{F})\right)^2 = (16.0000 + 7.3333)^2 = (23.3333)^2 = 544.4429.$$

Similarly, the denominator of $\hat{\varepsilon}$ equals

$$(a-1)\sum_{i=1}^{a-1}\sum_{j=1}^{a-1}(E_{ij}^*(\mathrm{F}))^2 = (3-1)\left[(16.0000)^2 + (-7.5055)^2 + (-7.5055)^2 + (7.3333)^2\right]$$
$$= 844.8847.$$

Thus the value of $\hat{\varepsilon}$ for these data is

$$\hat{\varepsilon} = \frac{544.4429}{844.8847} = 0.64444.$$

If we wanted, we could now use Equation 11.34 to find the corresponding value of $\tilde{\varepsilon}$:

$$\tilde{\varepsilon} = \frac{n(a-1)\hat{\varepsilon} - 2}{(a-1)[n-1-(a-1)\hat{\varepsilon}]}, \qquad \text{(11.34, repeated)}$$

which for our data yields $\tilde{\varepsilon} = .7276$.

What does Equation 13E3.1 for calculating $\hat{\varepsilon}$ from the **E*(F)** matrix reveal about the nature of the $\hat{\varepsilon}$ adjustment? First, notice that the adjusted test, unlike the unadjusted mixed-model test, incorporates information about the off-diagonal elements of **E*(F)** into the test. Because the off-diagonal elements are squared and appear only in the denominator of the expression for $\hat{\varepsilon}$ (see Equation 13E3.1), larger off-diagonal elements of **E*(F)** (either positive or negative) lead to lower values of $\hat{\varepsilon}$. This should seem reasonable, because remember that when homogeneity holds, the off-diagonal elements deviate from zero only because of sampling error. In this case, the off-diagonal elements should have values close to zero, minimizing their influence in the denominator. However, when homogeneity is not met, the off-diagonal elements may be nonzero even in the population. In this case, the off-diagonal elements may deviate appreciably from zero, causing $\hat{\varepsilon}$ to be much less than 1.0. Thus if homogeneity is violated, $\hat{\varepsilon}$ tends to compensate for the violation by reducing the degrees of freedom of the critical value.

Second, we saw in the previous section that if homogeneity holds, the diagonal elements of the population **E*(F)** matrix are equal to each other. Although it may not be immediately obvious from Equation 13E3.1, $\hat{\varepsilon}$ is also sensitive to the degree of inequality of the diagonal elements of **E*(F)**. This point can be understood most easily by comparing two hypothetical **E*(F)** matrices, both of which have off-diagonal elements of zero. Let's first consider such an **E*(F)** matrix that also has equal diagonal elements. For example, **E*(F)** might equal

$$\mathbf{E}^*(\mathbf{F}) = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}.$$

From Equation 13E3.1, $\hat{\varepsilon}$ for this matrix equals

$$\hat{\varepsilon} = \frac{(9+9)^2}{(3-1)(9^2+0^2+0^2+9^2)} = \frac{324}{2(81+81)} = 1.0.$$

Of course, we would expect $\hat{\varepsilon}$ to equal 1.0 when the homogeneity assumption is perfectly satisfied, as it is here. Now, however, let's consider another **E*(F)** matrix, which again has off-diagonal elements of zero, but this time has unequal diagonal elements. For example, **E*(F)** might equal

$$\mathbf{E}^*(\mathbf{F}) = \begin{bmatrix} 15 & 0 \\ 0 & 3 \end{bmatrix}.$$

From Equation 13E3.1, $\hat{\varepsilon}$ for this matrix equals

$$\hat{\varepsilon} = \frac{(15+3)^2}{(3-1)(15^2+0^2+0^2+3^2)} = \frac{324}{468} = 0.6923.$$

Thus unequal diagonal elements of **E*(F)** lower the value of $\hat{\varepsilon}$.

Using the $\hat{\varepsilon}$ adjustment (or $\tilde{\varepsilon}$) allows the mixed-model F test to be sensitive to the entire **E*(F)** matrix, removing a typical inadequacy of the unadjusted mixed-model test. When homogeneity holds, off-diagonal elements of **E*(F)** are near zero, and the diagonal elements are nearly equal to each other. As a result, $\hat{\varepsilon}$ (and $\tilde{\varepsilon}$) are close to 1.0, and the degrees of freedom for the critical value of the adjusted test are close to those of the unadjusted test. However, when homogeneity does not hold, off-diagonal elements of **E*(F)** are farther from zero, and/or the diagonal elements vary from each other in value. As a result, $\hat{\varepsilon}$ (and $\tilde{\varepsilon}$) may be substantially less than 1.0, lowering the degrees of freedom for the critical value of the adjusted test. The corresponding increase in the critical value itself prevents the increase in Type I errors that would occur with the unadjusted mixed-model test.

NOTES

1. Because of rounding error, multiplying a number in column 1 of Table 13E1.1 by 24 may not exactly reproduce the corresponding number in column 2. However, any discrepancy is due to the presentation of only three decimal places for the slope values. If enough decimal places were shown, the relationship would be exact.
2. The approach we used here of calculating a regression slope for each subject and then performing an analysis on the slopes has received considerable attention in the last few years as a general methodology

for handling complex problems in analyzing longitudinal data. Chapter 15 provides an introduction to this approach for analyzing longitudinal data.

REFERENCE: CHAPTER 13 EXTENSIONS

Huynh, H. (1978). Some approximate tests for repeated measurement designs. *Psychometrika*, *43*, 161–175.